

## Restricted Walks, Stability–Instability Transitions, and Dynamic Symmetries

F. T. Hioe<sup>1</sup>

Received July 14, 1989

---

The principal results from applying a transition matrix approach to the problem of self-avoiding walks which Domb and Hioe obtained some years ago are recalled. Some results are then presented on two other different physical problems: one on a universal critical exponent for a class of stability–instability transitions in the classical Hamiltonian systems, and the other on the existence of a characteristic set of constants of evolution when a quantum system possesses a certain type of symmetry. The similarity in some of the key concepts and methods used in these three problems, which involve studies of the distribution of the appropriate eigenvalues and the utilization of the existing symmetry, and the dissimilarity in some of the details are noted.

---

**KEY WORDS:** Restricted walks; critical exponent; stability–instability transitions; constants of evolution; Gell-Mann symmetry.

After graduating with a Bachelor of Science Degree in Physics from Imperial College London in 1963, I was privileged to be a graduate student of Prof. Cyril Domb at King's College London for a period of four years, 1963–1967. Professor Domb's spirit and style left an indelible imprint on me and my work. Together with the late Prof. Elliott Montroll, with whom I worked for several years as a postdoc, Prof. Domb had the strongest influence on my work with his ideas and approaches to physical problems. As a tribute to Prof. Domb on his retirement from Bar-Ilan University, I will first recall and describe briefly the problem of finite restricted random walk which he and I first worked on.<sup>(1–3)</sup> The study of this problem started with his ideas of examining the distribution of *eigenvalues* of the appropriate transition matrices, of simplifying the problem by taking into

---

<sup>1</sup> Department of Physics, St. John Fisher College, Rochester, New York 14618, and Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627.

account its *symmetry* using group theory, and of understanding how the fractional *critical indices* arise. I will then describe my more recent work on two seemingly quite different physical problems. One concerns the problem of stability–instability transitions in the classical Hamiltonian systems, and the other concerns the problem of finding constants of evolution in a quantum system and its relation to the problem of classification of elementary particles. A common feature will be seen to be present in all these problems which is related to the ideas and concepts underlined above which I learned from Prof. Domb.

## 1. TRANSITION MATRIX APPROACH TO SELF-AVOIDING WALKS

We considered a restricted walk of order  $r$  on a lattice which is defined as a random walk in which polygons with  $r$  vertices or less are excluded. We want to understand how the sequence of restricted walks approaches a self-avoiding walk in the limit that both  $r$  and the number of steps  $n$  approach infinity. A restricted walk with a finite  $r$  is a Markovian process and a transition matrix method for its study was initiated by Montroll<sup>(4)</sup> and pursued further by Fisher and Sykes.<sup>(5)</sup> To construct the transition matrix or the recurrence relations for a restricted walk of order  $r$ , we start with all possible  $(r-1)$ -step self-avoiding walks, denoted by  $\mathbf{w}_{r-1} = (w_{r-1}(1), w_{r-1}(2), \dots, w_{r-1}(c_{r-1}))$ , where  $c_{r-1}$  is the total number of  $(r-1)$ -step self-avoiding walks. The addition of a further step to a walk  $w_{r-1}(1)$  in all possible ways leads either to some forbidden self-intersections or to a walk of type  $w_r(k)$ , or type  $w_r(l)$ , etc. Considering in a similar way the addition of a step to the types  $w_{r-1}(2), w_{r-1}(3), \dots$  leads to a set of recurrence relations which can be conveniently expressed by the matrix equation

$$\mathbf{w}_r = \hat{A} \mathbf{w}_{r-1} \quad (1.1)$$

where the transition matrix  $\hat{A}$  is of order  $N \equiv c_{r-1}$ . The numbers of  $n$ -step walks ending in types  $w(1), w(2), \dots$  are then given by the components of  $\mathbf{w}_n$  in

$$\mathbf{w}_n = \hat{A}^{n-r+1} \mathbf{w}_{r-1} \quad (1.2)$$

These recurrence relations can be solved in a standard way. Let  $\lambda_{1r}, \lambda_{2r}, \dots, \lambda_{Nr}$  be the eigenvalues of the transition matrix  $\hat{A}$ ; then the total number of  $n$ -step restricted walks of order  $r$  may be written as

$$c_{nr} = \sum_{i_r}^N a_{i_r} \lambda_{i_r}^n \quad (1.3)$$

where  $a_{ir}$  are constants which can be determined from the eigenvectors and the initial walk distribution.

If  $\lambda_{1r}$  is the eigenvalue of largest modulus, then as the number of steps  $n \rightarrow \infty$ , the first term in (1.3) will dominate. But if we let  $n$  and  $r$  tend to infinity simultaneously, since the size of the transition matrix increases rapidly with  $r$ , ignoring the rest of the eigenvalues may not be justifiable and we have to examine their contributions to  $c_{nr}$  more carefully.

We first note that the numerical problem can be simplified considerably by grouping the eigenvalues<sup>(3)</sup> according to the irreducible representations of the crystallographic point group (see, e.g., ref. 6) appropriate for the lattice considered. Only the eigenvalues corresponding to the identity representation contribute to the total number of walks  $c_{nr}$ . The following picture also emerges from our numerical data for restricted walks of increasing order  $r$  on various two- and three-dimensional lattices:

(a) The largest eigenvalue  $\lambda_{1r}$  of the transition matrix  $\hat{A}$  is always contained in the group of eigenvalues corresponding to the identity representation. It is always real, distinct, and well separated from the rest of the eigenvalues in the group, which are generally complex and generally distributed rather symmetrically about the origin.

(b) The contribution from  $\lambda_{1r}^n$  to  $c_{nr}$  accounts for over 99% of the total contribution even as  $n$  is as small as  $r$ , and the contribution increases due to the increasing symmetry of the distribution of the rest of the eigenvalues with increasing  $r$ .

(c) The largest eigenvalue satisfies approximately the relation

$$\lambda_{1r} \simeq \mu(1 + g/r) \tag{1.4}$$

and the origin of the  $n^g$  term in the asymptotic formula for the total number  $c_n$  of self-avoiding walks of  $n$  steps is to be found in

$$c_n \simeq \lambda_{11}\lambda_{12}\cdots\lambda_{1n} \simeq \mu^n \prod_{r=1}^n (1 + g/r) \simeq n^g \mu^n \tag{1.5}$$

where  $g$  depends only on the dimensionality of the lattice and is found to be close to  $1/3$  for the two-dimensional lattices and close to  $1/6$  for the three-dimensional lattices. We can now understand the significance of the  $n^{1/3}$  and  $n^{1/6}$  coefficients in the expression for the total number of self-avoiding walks  $c_n$ . A self-avoiding walk can only establish its true character after an infinite number of steps and the limiting value  $\mu$  corresponds to this equilibrium state. During the first  $r$  steps it has not reached equilibrium and the freedom in choosing a new step is

approximately  $\lambda_{1r}$ , corresponding to a walk of order  $r$ . Relations (1.5) represents an end effect and it is the combination of these end effects which gives rise to the factor  $n^g$  in  $c_n$ .

If we define the generating function

$$C(x) = \sum_{n=0}^{\infty} c_n x^n \quad (1.6)$$

where  $c_0 = 1$  and  $c_n$  is the number of  $n$ -step self-avoiding walks, the asymptotic behavior (see, e.g., ref. 7) of  $c_n$  given by Eq. (1.5) implies that as  $x$  approaches its "critical" value  $1/\mu$ , which depends on the lattice considered, the function  $C(x)$  behaves like

$$C(x) \sim \frac{\text{const}}{(1 - \mu x)^{1+g}} \quad (1.7)$$

where  $1 + g$  is the critical exponent or index associated with the critical point  $1/\mu$  of the function  $C(x)$ . Thus, the behavior of the sequence of the largest eigenvalue of the transition matrix  $\hat{A}$  of Eq. (1.1) determines the critical exponent of an associated function (1.6).

Professor Domb and I also studied the correlation in a restricted walk of increasing order  $r$ . In this case, another group of eigenvalues corresponding to what we called the maximal representation appropriate for the lattice must be considered. The second largest eigenvalue  $\lambda_{2r}$  of the transition matrix  $\hat{A}$  is always contained in this group, and it is always real and quite separated from the rest of the eigenvalues in this group, which are generally complex and quite symmetrically distributed about the origin. The correlation between the  $s$ th step  $\mathbf{u}_s$  and the  $(s+t)$ th step  $\mathbf{u}_{s+t}$  is found to be given to a good approximation by

$$\langle \mathbf{u}_s \cdot \mathbf{u}_{s+t} \rangle \sim \text{const} \cdot (\lambda_{2r}/\lambda_{1r})^t \quad (1.8)$$

Our numerical data suggested that the ratio  $\lambda_{2r}/\lambda_{1r}$  which characterizes the correlations between steps would, as  $r \rightarrow \infty$ , approach 1, which, as is known, indicates the onset of long-range order (see, e.g., ref. 8).

At the time and long after I did the above work, I was often very much intrigued by the beauty and intricacies of analyzing the problem using the transition matrix method, examining the distribution of eigenvalues, taking into account of the symmetry by group theory, and associating the results with the concept of critical indices and long-range order. As I describe the results of two rather different physical problems in the following sections, the reader will recognize quite a bit of similarity in the mathematical approach even though the details may appear quite different.

## 2. STABILITY-INSTABILITY TRANSITIONS IN HAMILTONIAN SYSTEMS OF $n$ DIMENSIONS

In this section, we consider the stability of a class of simple periodic motions in a general Hamiltonian system of  $n$  dimensions.<sup>(9,10)</sup> If  $C$  is an adjustable parameter of the Hamiltonian, then the simple periodic motion may undergo stability-instability transitions (see, e.g., ref. 11) at several or infinitely many “critical values”  $C_p$ ,  $p = 1, 2, 3, \dots$ , of  $C$ . We want to show that the behavior of the largest Lyapunov exponent  $\mu$ , to be defined in the following, as  $C$  approaches  $C_p$  from the unstable region is given by

$$\mu = \text{const} \times |C - C_p|^\beta \tag{2.1}$$

where  $\beta = 1/2$ , independent of the transition point, type of transitions, or the dimensionality of the system.

Consider a general Hamiltonian system whose Hamiltonian is given by

$$H = \frac{1}{2} \sum_{j=1}^n m_j \dot{x}_j^2 + V(x_1, x_2, \dots, x_n) \tag{2.2}$$

where the potential energy  $V$  depends on the position coordinates only. The shape of the potential function can be quite arbitrary, but we are interested in those motions which are bounded; they may be regular or chaotic. The equations of motion are given by

$$m_j \ddot{x}_j + \partial V / \partial x_j = 0, \quad j = 1, 2, \dots, n \tag{2.3}$$

which are generally nonlinear. We assume that, under a set of initial conditions, the system has a simple periodic solution with a determinable real period  $\tau$ . An example in which this situation often occurs is when the initial condition is given by

$$x_j(0) = a, \quad \dot{x}_j(0) = 0, \quad x_k(0) = \dot{x}_k(0) = 0 \quad \text{for } k \neq j \tag{2.4}$$

for which the equations of motion (2.3) are assumed to give a solution of the form

$$x_j(t) = \phi(t), \quad x_k(t) = 0 \quad \text{for } k \neq j \tag{2.5}$$

where  $\phi(t) = \phi(t + \tau)$  is a periodic function with a period  $\tau$ . We shall consider the stability of this solution when the initial condition given by Eq. (2.4) is slightly changed. Let

$$\mathbf{w}(t) = \text{col}(\Delta x_1, \Delta \dot{x}_1, \Delta x_2, \Delta \dot{x}_2, \dots, \Delta x_n, \Delta \dot{x}_n) \tag{2.6}$$

be a  $2n$ -dimensional column vector whose components represent small perturbations  $\Delta x_r$  and  $\Delta \dot{x}_r$  from  $x_r$  and  $\dot{x}_r$ ,  $r=1, 2, \dots, n$ . The linearized equation of motion for  $\mathbf{w}(t)$  can be written as

$$\dot{\mathbf{w}} = \hat{A}(t)\mathbf{w} \quad (2.7)$$

where the transition matrix  $\hat{A}(t)$  can be written in an explicitly time-dependent form in terms of  $\phi(t)$  because of the solution (2.5). Thus  $\hat{A}(t)$  is periodic with a period

$$T = \tau \quad \text{or} \quad \tau/2 \quad (2.8)$$

the latter being the case if the period of  $\phi^2(t)$  is  $\tau/2$  and only even powers of  $\phi(t)$  appear in  $\hat{A}(t)$ . The dimension of  $\hat{A}(t)$  is generally  $2n$ . In practice, however, it often happens that certain symmetries or simplifying features can be used to reduce  $\hat{A}(t)$  to a smaller size. The literature on the stability analysis of the type of equations given by Eq. (2.7) pioneered by Lyapunov is very extensive (see, e.g., ref. 11).

Let the column vectors  $\mathbf{w}_k(t)$ ,  $k=1, 2, \dots, 2n$ , be the  $2n$  solutions of Eq. (2.7) corresponding to the initial values given by  $w_{jk}(0) = \delta_{jk}$ , where  $w_{jk}(t)$  is the  $j$ th component of  $\mathbf{w}_k(t)$ . Consider the matrix  $\hat{W}(t)$  whose elements  $w_{jk}(t)$  are the components of these  $2n$  fundamental solutions  $\mathbf{w}(t)$  of Eq. (2.7). Since  $\hat{A}(t)$  is periodic with period  $T$ , there exists a nonsingular constant matrix  $\hat{P}$  such that

$$\hat{W}(t+T) = \hat{W}(t)\hat{P} \quad (2.9)$$

Setting  $t=0$  and noting that  $\hat{W}(0)$  is a unit matrix, it follows that the elements  $p_{jk}$  of  $\hat{P}$  are given by

$$p_{jk} = w_{jk}(T) \quad (2.10)$$

The matrix  $\hat{P}$ , which can be readily numerically determined from Eq. (2.10), plays a fundamental role in the stability analysis, for it can be shown, using a similar argument to that which led to the Floquet theorem, that the eigenvalues  $\lambda_j$ ,  $j=1, 2, \dots, 2n$ , of  $\hat{P}$  determine the stability or instability of the system whose linearized equations of motion for the small perturbations are given by Eq. (2.7). More specifically, denoting

$$\lambda_j = \exp(\mu_j T) \quad (2.11)$$

or

$$\mu_j = T^{-1} \ln \lambda_j \quad (2.12)$$

the general solution of the differential equation (2.7) is given by

$$\mathbf{w}(t) = \sum_{j=1}^{2n} c_j e^{\mu_j t} \boldsymbol{\Psi}_j(t) \tag{2.13}$$

where the  $c_j$  are constants and the  $\boldsymbol{\Psi}_j(t)$  are functions which are periodic with the period  $T$ . Notice that we have extracted the essential part of the eigenvalues of  $\hat{A}(t)$  in (2.7), namely the eigenvalues of  $\hat{P}$  given by (2.9) and (2.10), for our study.

The eigenvalue equation of the matrix  $\hat{P}$  can be shown<sup>(9)</sup> to be reciprocal, i.e., it is of the form

$$\lambda^{2n} + \alpha_1 \lambda^{2n-1} + \alpha_2 \lambda^{2n-2} + \dots + \alpha_{2n-2} \lambda^2 + \alpha_{2n-1} \lambda + 1 = 0 \tag{2.14}$$

where  $\alpha_1 = \alpha_{2n-1}$ ,  $\alpha_2 = \alpha_{2n-2}$ , etc. That is to say, for every eigenvalue  $\lambda$  of Eq. (2.14), there is also the eigenvalue  $\lambda^{-1}$ . The stable region is characterized by the roots  $\lambda_j$  distributed over the unit circle in the complex plane, and the unstable region is characterized by one or more of these eigenvalues having an absolute value greater than 1. The transition from a stable to an unstable region can be classified according to three types: (I) a real eigenvalue crosses the unit circle at +1; (II) two conjugate eigenvalues cross the unit circle simultaneously; and (III) a real eigenvalue crosses the unit circle at -1.

It is always possible to write the characteristic equation (2.14) in the form

$$(\lambda^2 - a_1 \lambda + 1)(\lambda^2 - a_2 \lambda + 1) \dots (\lambda^2 - a_n \lambda + 1) = 0 \tag{2.15}$$

where  $a_1, a_2, \dots, a_n$  can be expressed as roots of an  $n$ th-degree algebraic equation whose coefficients can be determined recursively from the coefficients  $\alpha_1, \alpha_2, \dots, \alpha_{2n-1}$  of Eq. (2.14). In the stable region, all the  $a$ 's are real and have absolute values  $\leq 2$ . A transition of type I to an unstable region as a result of changing the value of a parameter of the system past its critical value is characterized by one of the  $a$ 's,  $a_j$  say, crossing the value +2 to a value greater than +2, while the remaining  $a$ 's remain real and  $\leq 2$  in absolute values. Thus, a transition of type I from a stable to an unstable region is characterized by a complex-conjugate pair of roots on the unit circle approaching each other and closing in on the positive real axis, becoming degenerate at the value +1 at the stability-instability transition point, and becoming separate again but appearing on two sides of +1 on the real axis, their values remaining reciprocal of each other. Similarly, a transition of type III is characterized by one of the  $a$ 's in Eq. (2.15) crossing the value -2 to a value less than -2, which implies a complex-conjugate

pair of roots closing in on the negative real axis and becoming degenerate at the value  $-1$  at the transition point, and then becoming separate and appearing on two sides of  $-1$  on the real axis. In either case, the roots of the equation  $\lambda^2 - a_j\lambda + 1 = 0$  given by

$$\lambda_{\pm} = \frac{1}{2}[a_j \pm (a_j^2 - 4)^{1/2}] \quad (2.16)$$

change from complex to real, but all the  $a$ 's in Eq. (2.15) remain real and thus remain analytic functions of the parameters of the system, since all the real coefficients of Eq. (2.14) are analytic functions of the parameters of the system. Thus, as the parameter  $C$  approaches the transition point  $C_p$  from the unstable region,  $a_j$  can be written as  $a_j = \pm 2 \pm \varepsilon$ , where the positive and negative signs refer to transitions of types I and III, respectively, and where  $\varepsilon = \text{const} \cdot |C - C_p| \geq 0$ . Thus we find, from Eq. (2.16), that the largest roots in absolute values are given, respectively, in type I and type III transitions, by  $\lambda = \pm 1 \pm \varepsilon^{1/2}$ , or, as  $C \rightarrow C_p$  from the unstable region, the behavior of the largest Lyapunov exponent is given, from Eq. (2.12), by Eq. (2.1), with  $\beta = 1/2$  independent of the transition points or the dimensionality of the system. We shall not consider the case in which, by an accident or a symmetry, the coefficient of  $|C - C_p|^{1/2}$  happens to be equal to zero.

A transition of type II from a stable to unstable region is characterized by two of the  $a$ 's,  $a_j$  and  $a_{j+1}$ , say, in Eq. (2.15) changing from real to a complex-conjugate pair, while the remaining  $a$ 's remain real and  $< 2$  in absolute values. Since  $a_j$  and  $a_{j+1}$  change from real to complex values across the transition point, neither of them is an analytic function of the parameters of the system generally. Thus we cannot, for example, assume that  $a_j = a_{j0} + \varepsilon$  in the neighborhood of the transition point. On the other hand, we have

$$(\lambda^2 - a_j\lambda + 1)(\lambda^2 - a_{j+1}\lambda + 1) = \lambda^4 + A\lambda^3 + B\lambda^2 + A\lambda + 1 \quad (2.17)$$

where

$$A = -(a_j + a_{j+1}), \quad B = a_j a_{j+1} + 2 \quad (2.18)$$

From Eqs. (2.17) and (2.18), we get

$$\begin{aligned} a_j &= -\frac{1}{2}[A + (A^2 - 4B + 8)^{1/2}] \\ a_{j+1} &= -\frac{1}{2}[A - (A^2 - 4B + 8)^{1/2}] \end{aligned} \quad (2.19)$$

and

$$\lambda_{\pm} = \frac{1}{2}[a \pm i(4 - a^2)^{1/2}] \quad (2.20)$$

where  $a$  denotes  $a_j$  or  $a_{j+1}$ . We note that since  $a_j$  and  $a_{j+1}$  are complex conjugate,  $A$  and  $B$  are always real, and hence they are always analytic functions of the parameters of the system. Thus, in the neighborhood of the



transition point,  $A$  and  $B$  can be written as  $A = A_0 + \varepsilon$ ,  $B = B_0 + \varepsilon$ , and at the transition point,  $A_0^2 - 4B_0 + 8 = 0$ . As the parameter  $C$  approaches the transition point  $C_p$  from the unstable region,  $a_j = -\frac{1}{2}(A + i\varepsilon^{1/2})$ ,  $a_{j+1} = -\frac{1}{2}(A - i\varepsilon^{1/2})$ , from Eq. (2.19). Substituting these into Eq. (2.20), we find  $\lambda_{\pm} = \frac{1}{4}[-A_0 \pm i(16 - A_0^2)^{1/2}] \pm c\varepsilon^{1/2}$ , where  $c$  is some complex constant. Thus, the absolute square of the largest root is given by  $|\lambda|^2 = 1 + \varepsilon^{1/2}$ , where  $\varepsilon$  is equal to some positive constant times  $|C - C_p|$ . Using Eq. (2.12), we find exactly the same behavior given by Eq. (2.1) for the Lyapunov exponent as that for the type I and type III transitions, with the same critical exponent  $\beta = \frac{1}{2}$ , which is independent of the transition point, type of transitions, or the dimensionality of the system.

There is a close analogy between the behavior of the Lyapunov exponent, Eq. (2.1), with the behavior of the long-range order (e.g., magnetization in a ferromagnet) in critical phenomena in statistical mechanics. The exponent  $\beta = \frac{1}{2}$  corresponds to the mean field result. The distribution of the eigenvalues  $\lambda_j$  and its behavior as the stability–instability transition point of type I or III is approached also have their analogs in thermodynamic phase transitions in the Lee–Yang theorem<sup>(12)</sup> on the distribution of roots of the grand partition function. However, the behavior of the eigenvalues corresponding to the stability–instability transition of type II does not appear to have any analog in equilibrium thermodynamics.

### 3. DYNAMIC SYMMETRIES AND GROUPING OF PRODUCTS OF WAVEFUNCTIONS

In this section, we consider a particular type of symmetry which may arise in the dynamical evolution of a quantum system consisting of  $N$  transition states when it interacts with an external field. The probability amplitudes of the system represented by the  $N$  components of the column wavevector

$$\Psi(t) = \text{col}(\Psi_1(t), \Psi_2(t), \dots, \Psi_N(t))$$

obey the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}(t)\Psi \tag{3.1}$$

where  $\hat{H}(t)$  is an  $N \times N$  Hermitian, generally time-dependent Hamiltonian matrix of the system. As in the problems expressed by Eqs. (1.1) and (2.7) in the previous sections, the interest in the problem may not simply lie with finding the solutions of these differential equations for a particular situation, but in seeing what unusual properties these equations might have under certain variable physical conditions. Questions which are of interest to us here are the following:

(1) Aside from the obvious conservation of total population given by

$$\sum_{n=1}^N |\Psi_n(t)|^2 = \text{const} \quad (3.2)$$

what other constants of evolution may exist?

(2) Can we construct a suitable unitary transformation by a time-independent unitary matrix  $\hat{U}$  such that in the Schrödinger equation for the transformed wavevector  $\Psi(t) = \hat{U}^\dagger \Psi(t)$  given by

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{\mathcal{H}}(t) \Psi \quad (3.3)$$

the transformed Hamiltonian

$$\hat{\mathcal{H}}(t) = \hat{U}^\dagger \hat{H}(t) \hat{U} \quad (3.4)$$

becomes a block diagonal form? If so, the components of the transformed wavevector  $\Psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_N(t))$  can be grouped into various independent subsets such that components belonging to different subsets evolve independently of each other.

(3) Given that  $\Psi(t)$  can be grouped into various independent subsets, how do products of the components, such as  $\psi_j^*(t) \psi_k(t)$ , which may be of more physical significance, fall into groups?

An  $N$ -state quantum system whose dynamical evolution is given by (3.1) is said to possess a certain dynamic symmetry if the system possesses one or more constants of evolution other than (3.2). It is said to possess what I called the Gell-Mann dynamic symmetry<sup>(13)</sup> if a time-independent unitary matrix  $\hat{U}$  can be found such that the transformed Hamiltonian  $\hat{\mathcal{H}}(t)$  of Eq. (3.3) is a block diagonal matrix of the form

$$\hat{\mathcal{H}}(t) = -\hbar \begin{bmatrix} 0 & h_{1,2}(t) & | & & | & & | & & \\ h_{1,2}^*(t) & \Delta(t) & | & 0 & | & 0 & | & 0 & \\ \hline & 0 & | & 0 & | & & | & 0 & \\ & & | & & \Delta(t) & | & 0 & & \\ \hline & 0 & | & 0 & | & 0 & | & & \\ & & | & & & \Delta(t) & | & 0 & \\ \hline & 0 & | & 0 & | & 0 & | & \cdot & \\ & & | & & & & | & \cdot & \end{bmatrix} \quad (3.5)$$

A particularly interesting class of physical Hamiltonians  $\hat{H}(t)$  which possess the Gell-Mann symmetry has its matrix elements  $H_{jk}(t)$  given as follows:

For  $j \neq k$ ,

$$H_{jk}(t) = \begin{cases} 0, & |j-k| = \text{even} & (3.6a) \\ a_j a_k^* f(t), & |j-k| = \text{odd and } j \text{ odd} & (3.6b) \\ a_j a_k^* f^*(t), & |j-k| = \text{odd and } j \text{ even} & (3.6c) \end{cases}$$

and

$$H_{jj}(t) = \begin{cases} 0, & j \text{ odd} & (3.7a) \\ g(t), & j \text{ even} & (3.7b) \end{cases}$$

where  $j, k = 1, 2, \dots, N$ ,  $f(t)$  and  $g(t)$  are any arbitrary time-dependent functions, and  $a_j$  are any arbitrary constants. The special features expressed by Eqs. (3.6a) and (3.7a) occur in many physical problems. For example, in quantum electronics involving a system of atoms or molecules with  $N$  transition levels interacting with a laser field, Eq. (3.6a) is automatically satisfied because of the electric dipole selection rule, and Eq. (3.7a) would be satisfied if the system is operated under the so-called two-photon resonance condition.<sup>(13)</sup> The matrix elements  $A_{jk}(t)$  of  $\hat{A}(t)$  in Eq. (2.7) also possess similar properties as expressed by Eqs. (3.6a) and (3.7a). Equations (3.6b), (3.6c), and (3.7b) are the special features required for the system to possess the Gell-Mann symmetry. The process of proving<sup>(13,14)</sup> that the  $\hat{H}(t)$  given by Eqs. (3.6) and (3.7) can be transformed into the  $\hat{\mathcal{H}}(t)$  given by Eq. (3.5) rests on the construction of a time-independent unitary matrix  $\hat{U}$  whose columns consist of the orthonormalized eigenvectors of a matrix  $\hat{H}^{(0)}$  whose matrix elements are given by

$$H_{jk}^{(0)} = \begin{cases} a_j a_k^*, & |j-k| = \text{odd} \\ 0, & |j-k| = \text{even} \end{cases} \quad (3.8)$$

Notice that here the eigenvalues and eigenvectors of  $\hat{H}(t)$  are not the quantities of direct relevance. We have, on the other hand, “extracted” that part of  $\hat{H}(t)$  given by  $\hat{H}^{(0)}$  and used its eigenvectors to transform  $\hat{H}(t)$  into a characteristic form  $\hat{\mathcal{H}}(t)$  given by (3.5) for which Eq. (3.3) holds.

An important feature of a system possessing the Gell-Mann symmetry is that it has the following characteristic set of constants of evolution:

$$\begin{aligned} |\mathbf{u}_1^\dagger \cdot \Psi(t)|^2 + |\mathbf{u}_2^\dagger \cdot \Psi(t)|^2 &= \text{const} \\ |\mathbf{u}_m^\dagger \cdot \Psi(t)| &= \text{const}, \quad m = 3, 4, \dots, N \end{aligned}$$

where  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N$  are the column vectors which make up the matrix  $\hat{U}$ , or

$$|\psi_1(t)|^2 + |\psi_2(t)|^2 = \text{const}$$

$$|\psi_m(t)| = \text{const}, \quad m = 3, 4, \dots, N$$

in terms of the components of the transformed wavevector  $\boldsymbol{\psi}(t)$ . This characteristic set of constants of motion is seen to closely resemble the set of quantum numbers associated with the isospin invariance, strangeness, charm, bottom, top, etc., in elementary particle physics.

**Table I. The Grouping of the Wavefunctions for the "Pseudoscalar Mesons" (Upper Sign) and for the "Vector Mesons" (Lower Sign)<sup>a</sup>**

One group of three:

$$2^{-1/2}(\psi_1\bar{\psi}_2 \pm \bar{\psi}_2\psi_1),$$

$$\frac{1}{2}[(\psi_1\bar{\psi}_1 - \psi_2\bar{\psi}_2) \pm (\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2)],$$

$$2^{-1/2}(\bar{\psi}_1\psi_2 \pm \psi_2\bar{\psi}_1)$$

$N - 2$  groups of four:

$$2^{-1/2}(\psi_1\bar{\psi}_m \pm \bar{\psi}_m\psi_1), 2^{-1/2}(\psi_2\bar{\psi}_m \pm \bar{\psi}_m\psi_2),$$

$$2^{-1/2}(\bar{\psi}_1\psi_m \pm \psi_m\bar{\psi}_1), 2^{-1/2}(\bar{\psi}_2\psi_m \pm \psi_m\bar{\psi}_2)$$

$\frac{1}{2}(N - 2)(N - 3)$  groups of two:

$$2^{-1/2}(\psi_m\bar{\psi}_n \pm \bar{\psi}_n\psi_m), 2^{-1/2}(\bar{\psi}_m\psi_n \pm \psi_n\bar{\psi}_m), m \neq n$$

$N - 2$  groups of one:

$$12^{-1/2}[(\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2 - 2\psi_3\bar{\psi}_3) \pm (\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 - 2\bar{\psi}_3\psi_3)],$$

$$24^{-1/2}[(\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2 + \psi_3\bar{\psi}_3 - 3\psi_4\bar{\psi}_4) \pm (\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 + \bar{\psi}_3\psi_3 - 3\bar{\psi}_4\psi_4)],$$

...

$$[2N(N - 1)]^{-1/2} \left\{ \left[ \sum_{j=1}^{N-1} \psi_j\bar{\psi}_j - (N - 1)\psi_N\bar{\psi}_N \right] \pm \left[ \sum_{j=1}^{N-1} \bar{\psi}_j\psi_j - (N - 1)\bar{\psi}_N\psi_N \right] \right\}$$

For the "vector meson," we have an additional member:

$$(2N)^{-1/2} \sum_{j=1}^N (\psi_j\bar{\psi}_j - \bar{\psi}_j\psi_j) \text{ as a member of a group of one}$$

Singlet:

$$(2N)^{-1/2} \sum_{j=1}^N (\psi_j\bar{\psi}_j + \bar{\psi}_j\psi_j)$$

<sup>a</sup>  $m, n = 3, 4, \dots, N$ .

If we now form products  $\bar{\psi}_j\psi_k$  and  $\psi_j\psi_k\psi_l$  of the components of this wave vector, where the components of different subscripts need not commute, and form various symmetrized and antisymmetrized combinations of them and examine how they can be grouped into independent subsets based on their dynamics given by Eqs. (3.3) and (3.5), we find<sup>(15)</sup> that the groupings of these product wavefunctions agree precisely with the groupings of the corresponding elementary particles: the pseudoscalar and vector mesons, and the baryons. More precisely, for  $N$  possible transition states or “quark flavors,” the way the various combinations of  $\bar{\psi}_j\psi_k$  or the “mesons” and the various combinations of  $\psi_j\psi_k\psi_l$  or the “baryons” fall into independent subsets according to their dynamics is presented in Tables I and II, where the wavefunctions  $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \dots$  “correspond” to the wavefunctions of  $d, u, s, c, b, t, \dots$  quarks, respectively. In Table II,  $\sigma_n$  denotes

$$\sigma_n = \frac{1}{6}n(n-1)(n-2)$$

and the combinations denoted by the symbol  $(\psi_j\psi_k\psi_l)$  under the headings  $S$  (symmetric),  $M_S$  (mixed symmetric),  $M_A$  (mixed antisymmetric) and  $A$  (antisymmetric) have their meanings given in Table III.

**Table II. The Grouping of the “Baryons”<sup>a</sup>**

---

$S$ (total number = $\sigma_{N+2}$ ):
One group of four: $(\psi_1\psi_1\psi_1), (\psi_1\psi_1\psi_2), (\psi_1\psi_2\psi_2), (\psi_2\psi_2\psi_2)$
$N-2$ groups of three: $(\psi_1\psi_1\psi_m), (\psi_1\psi_2\psi_m), (\psi_2\psi_2\psi_m)$
$\frac{1}{2}(N-2)(N-1)$ groups of two: $(\psi_m\psi_n\psi_1), (\psi_m\psi_n\psi_2)$
$\frac{1}{6}(N-2)(N-1)N$ groups of one: $(\psi_m\psi_n\psi_p)$
$M_S$ or $M_A$ (total number = $2\sigma_{N+1}$ ):
One group of two: $(\psi_1\psi_1\psi_2), (\psi_2\psi_2\psi_1)$
$N-2$ groups of four: $(\psi_1\psi_1\psi_m), (\psi_1\psi_2\psi_m), (\psi_1\psi_2\psi'_m), (\psi_2\psi_2\psi_m)$
$(N-2)^2$ groups of two: $(\psi_m\psi_n\psi_1), (\psi_m\psi_n\psi_2)$
$\frac{1}{3}(N-3)(N-2)(N-1)$ groups of one: $(\psi_m\psi_n\psi_p)$
$A$ (total number = $\sigma_N$ ):
$N-2$ groups of one: $(\psi_1\psi_2\psi_m)$
$\frac{1}{2}(N-2)(N-3)$ groups of two: $(\psi_m\psi_n\psi_1), (\psi_m\psi_n\psi_2), m \neq n$
$\frac{1}{6}(N-2)(N-3)(N-4)$ groups of one: $(\psi_m\psi_n\psi_p), m \neq n \neq p$

---

<sup>a</sup>  $m, n, p = 3, 4, \dots, N$ .

**Table III. The Symmetric ( $S$ ), Mixed Symmetric ( $M_S$ ), Mixed Antisymmetric ( $M_A$ ), and Antisymmetric ( $A$ ) Arrangements of "Quark Flavor" Wavefunctions  $\alpha, \beta, \gamma$**

---

$S:$	$(\alpha\alpha\alpha) = \alpha\alpha\alpha$ $(\alpha\alpha\beta) = 3^{-1/2}(\alpha\beta\alpha + \beta\alpha\alpha + \alpha\alpha\beta)$ $(\alpha\beta\gamma) = 6^{-1/2}[(\beta\gamma + \gamma\beta)\alpha + (\gamma\alpha + \alpha\gamma)\beta + (\alpha\beta + \beta\alpha)\gamma]$
$M_S:$	$(\alpha\alpha\beta) = 6^{-1/2}[(\alpha\beta + \beta\alpha)\alpha - 2\alpha\alpha\beta]$ $(\alpha\beta\gamma) = 12^{-1/2}[(\beta\gamma + \gamma\beta)\alpha + (\gamma\alpha + \alpha\gamma)\beta - 2(\alpha\beta + \beta\alpha)\gamma]$ $(\alpha\beta\gamma)' = (1/2)[(\beta\gamma + \gamma\beta)\alpha - (\gamma\alpha + \alpha\gamma)\beta]$
$M_A:$	$(\alpha\alpha\beta) = 2^{-1/2}(\alpha\beta - \beta\alpha)\alpha$ $(\alpha\beta\gamma) = (1/2)[-(\beta\gamma - \gamma\beta)\alpha + (\gamma\alpha - \alpha\gamma)\beta]$ $(\alpha\beta\gamma)' = 12^{-1/2}[-(\beta\gamma - \gamma\beta)\alpha - (\gamma\alpha - \alpha\gamma)\beta + 2(\alpha\beta - \beta\alpha)\gamma]$
$A:$	$(\alpha\beta\gamma) = 6^{-1/2}[(\beta\gamma - \gamma\beta)\alpha + (\gamma\alpha - \alpha\gamma)\beta + (\alpha\beta - \beta\alpha)\gamma]$

---

Inclusion of the spin would not affect the grouping of these product wavefunctions if we assume that the matrix elements of  $\hat{\mathcal{H}}(t)$  in (3.3) are independent of the spin.

It is tempting to suggest that a Hamiltonian of the form (3.5) could be the low-energy limit of a more fundamental relativistic field theory in the context of elementary particle physics. Regardless of the origin of these wavefunctions, we have found the Gell-Mann-type symmetry to be useful in many physical problems.

#### 4. SUMMARY

A unifying feature in the three problems discussed in this paper is contained in the form of the starting equations (1.1), (2.7), and (3.1). The problems of analyzing the distribution of the appropriate eigenvalues and finding critical exponents occur in the first two problems, and the problem of taking advantage of the special symmetry occurs in all three. I hope to have shown some unexpected physical richness displayed by these examples, and I am forever thankful to Prof. Domb for first showing me the way to find and appreciate these treasures.

#### ACKNOWLEDGMENTS

This research is supported by the U.S. Department of Energy, Office of Basic Energy Sciences, Division of Chemical Sciences, under grant DE-FG02-84ER13243.

## REFERENCES

1. F. T. Hioe, Ph.D. Thesis, University of London (1967), unpublished.
2. C. Domb and F. T. Hioe, *J. Phys. C* **3**:2223 (1970).
3. F. T. Hioe and C. Domb, *J. Stat. Phys.* **10**:183 (1974).
4. E. W. Montroll, *J. Chem. Phys.* **18**:734 (1950).
5. M. E. Fisher and M. F. Sykes, *Phys. Rev.* **114**:45 (1959).
6. M. Hamermesh, *Group Theory and Its Application* (Addison-Wesley, Reading, Massachusetts, 1962).
7. C. Domb and M. S. Green, eds., *Phase Transitions and Critical Phenomena*, Vol. 6 (Academic Press, New York, 1976).
8. C. Domb, *Adv. Phys.* **9**:149 (1960); G. F. Newell and E. W. Montroll, *Rev. Mod. Phys.* **25**:353 (1953).
9. F. T. Hioe and Z. Deng, *Phys. Rev. A* **35**:847 (1987).
10. F. T. Hioe, *Phys. Rev. A* **39**:2628 (1989).
11. W. Hahn, *Stability of Motion* (Springer-Verlag, New York, 1967); A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983); J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1983).
12. T. D. Lee and C. N. Yang, *Phys. Rev.* **87**:410 (1952).
13. F. T. Hioe, *J. Opt. Soc. Am. B* **5**:859 (1988); **6**:1245 (1989); *Phys. Rev. A* **28**:879 (1983); **32**:2824 (1985).
14. F. T. Hioe and C. E. Carroll, *Phys. Rev. A* **37**:3000 (1988).
15. F. T. Hioe, *J. Phys. A* **22**:4143 (1989).